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Symplectic Geometry

Homework 4

Exercise 1. (4 points)
Show that for the standard symplectic structure ω_0 on $\mathbb{R}^{2n}$ and for any $A \in Gl(2n, \mathbb{R})$ it holds that

$$A^T J_0 A = J_0 \Leftrightarrow \omega_0(Au, Av) = \omega_0(u, v) \text{ for all } u, v \in \mathbb{R}^{2n}.$$

Exercise 2. (5 points)
Write $A \in Gl(2n)$ as a block matrix

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

with $A_j \in Mat(n \times n; \mathbb{R})$. Translate the condition $A \in Sp(2n)$ into conditions on A_j's.

Exercise 3. (4 points)
Show that if $A \in Sp(2n)$ then $A^T \in Sp(2n)$.

Exercise 4. (4 points)
Recall the map $\Psi\colon Gl(n, \mathbb{C}) \to Gl(2n, \mathbb{R})$ defined by $X + iY \mapsto \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}$. Show that for $A \in Gl(2n, \mathbb{R})$ it holds that

$$A \in \Psi(Gl(n, \mathbb{C})) \Leftrightarrow AJ_0 = J_0 A.$$

Exercise 5. (4 points)
Show that if $A \in O(2n) \cap Sp(2n)$ then $A \in \Psi(U(n))$.

Exercise 6. (4 points)
Show that if $A \in Sp(2n)$ then $(AA^T)^{-\frac{1}{2}} A$ is orthogonal. (The fact that $(AA^T)^{-\frac{1}{2}}$ is well defined was shown in class, and so you don't need to re-prove it.)

Exercise 7. (5 points)
Under what conditions on A does $(u, v) \mapsto u^T A v$ define an inner product? A symplectic form?

Exercise 8. (5 points)
For X, Y $n \times n$ real matrices, compare $\det(X + iY) \in \mathbb{C}$ and $\det \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \in \mathbb{R}$.

Exercise 9. (5 points)
Fix n integers: $k_1, \ldots, k_n$. Find the Maslov index of a loop given by

$$\mathbb{R}/\mathbb{Z} \ni t \mapsto \begin{bmatrix} \begin{matrix} \cos(2\pi k_1 t) & \sin(2\pi k_1 t) \\ -\sin(2\pi k_1 t) & \cos(2\pi k_1 t) \end{matrix} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \begin{matrix} \cos(2\pi k_n t) & \sin(2\pi k_n t) \\ -\sin(2\pi k_n t) & \cos(2\pi k_n t) \end{matrix} \end{bmatrix} \in Sp(2n)$$

(i.e. the counterclockwise rotation of each $\mathbb{R}^2$ in $\mathbb{R}^{2n}$ with speeds $k_1, \ldots, k_n$, respectively).

Hand in: Thursday November 17th
in the exercise session
in Übungsraum 1, MI